Learning the dynamics of nonlinear systems with a certified output set through linear matrix inequalities

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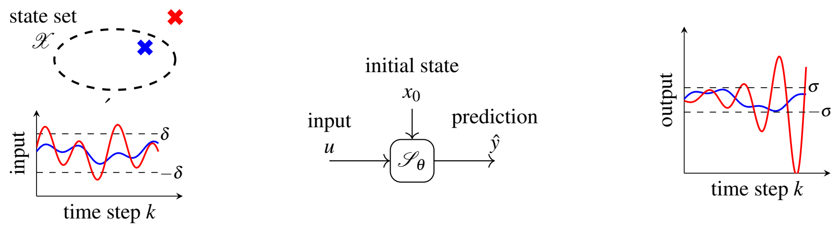

Figure 1: Given an input sequence $u \in \mathscr{U}_\delta$ and an initial state $x_0 \in \mathscr{X}$, our framework computes a certified output set $\mathscr{Y}_\sigma$ such that the model prediction $\hat{y}_k \in \mathscr{Y}_\sigma$ for all $k > 0$. This safety guarantee is illustrated by the blue trajectories, which remain within the prescribed bounds (black dashed lines). Conversely, the red trajectories represent cases where the input or initial state conditions are violated, leading to predictions that are not guaranteed to remain bounded.

1 Introduction

We address the problem of identifying nonlinear dynamical systems using Recurrent Neural Network (RNN) architectures. While RNNs excel at capturing complex nonlinearities, they lack certificates to ensure safe model predictions under unknown inputs, a limitation that hinders their use in safety-critical applications. To bridge this gap, we employ tools from systems theory to derive linear matrix inequality constraints (LMI) on the network's learnable parameters. When these constraints are satisfied, we provide a certificate that the model's predictions remain within a compact output set for inputs and initial states from a predefined input and state set. This behavior is illustrated in Figure 1.

2 Certified output set

We consider nonlinear state space models of the form

$$\begin{pmatrix} x_{k+1} \\ \hat{y}_k \\ v_k \end{pmatrix} = \begin{pmatrix} A & B & B_2 \\ C & D & D_{12} \\ C_2 & D_{21} & 0 \end{pmatrix} \begin{pmatrix} x_k \\ u_k \\ w_k \end{pmatrix} \tag{1}$$

with $w_k = \Delta(z_k)$ and learnable model parameters $\theta = \{A, B, B_2, C, D, D_{12}, C_2, D_{21}\}$. The nonlinear function Δ, called the activation function, is the *deadzone* nonlinearity, which satisfies the generalized sector conditions [1]. Based on these conditions, we derive LMI constraints on θ which guarantee that the predicted output $\hat{y}$ is certified to stay in a compact set, provided that the input sequence and initial conditions are taken from predefined sets. These conditions are enforced during training by regularizing the loss function with logarithmic barrier functions. While globally stable models cannot recover the red trajectories in Figure 1, our architecture can recover both the red and the blue trajectories, thereby extending the class of learnable models. Identification methods that do not provide stability certificates do not provide the bounds indicated by the black dashed lines.

Table 1: Evaluation results on the *Silverbox* dataset.

Model	1000 RMSE$(y, \hat{y})$ in mV	σ in mV	#Par.
GenSec (ours)	[5.72; 9.38; 5.50]	3.39	653
NoSec	[1.67; 2.21; 1.287	-	553
SUBNET [2]	[0.36; 1.4; 0.32]	-	-

3 Experiments

We evaluate the proposed learning methodology on the *Silverbox* benchmark, providing a certified output set, $|\hat{y}| \leq \sigma$, for $k \geq 0$[1]. We compare the proposed model GenSec (ours) with an unconstrained RNN model of the form (1) (NoSec) as well as an unconstrained baseline model, SUBNET [2]. While unconstrained models achieve higher prediction accuracy on the test set, they do not provide a certified output set. Our preliminary results indicate that, although our approach does not yet reach the accuracy of unconstrained models, it provides the additional guarantee that model predictions remain inside a certified set. For the *Silverbox* dataset, this corresponds to a physically interpretable output interval measured in mV.

References

[1] J. G. Da Silva and S. Tarbouriech, "Antiwindup design with guaranteed regions of stability: an LMI-based approach," *IEEE Transactions on Automatic Control*, vol. 50, no. 1, pp. 106–111, 2005.

[2] G. I. Beintema, M. Schoukens, and R. Tóth, "Deep subspace encoders for nonlinear system identification," *Automatica*, vol. 156, p. 111210, 2023.

[1] `https://www.nonlinearbenchmark.org/benchmarks/silverbox`