University of Stuttgart
Institute for Artificial Intelligence
Analytic Computing

ANALYTIC COMPUTING

SimTech
Cluster of Excellence

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Learning the dynamics of nonlinear systems with regional stability guarantees through LMI constraints

Daniel Frank, Fahim Shakib, Steffen Staab

Introduction

Learning the dynamics of nonlinear systems with regional stability guarantees through LMI constraints

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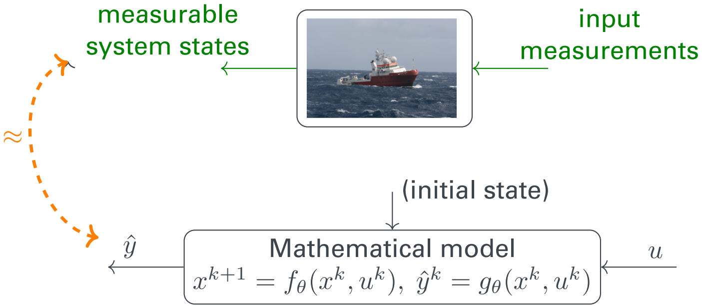
Figure 1: Learning a mathematical model from observations of an unknown system.

Learning the dynamics of nonlinear systems with regional stability guarantees through LMI constraints



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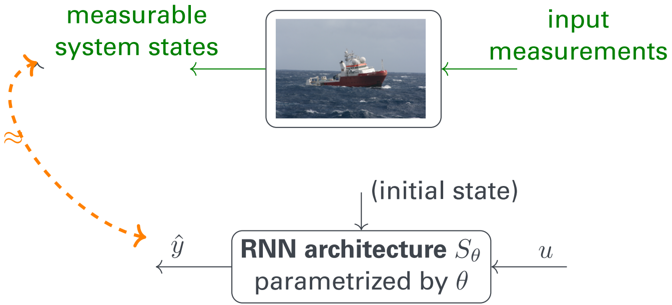

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Linear Matrix Inequalities (LMI) Scherer and Weiland [3]

With the decision vector $\theta = (\theta_1 \cdots \theta_n)^T \in \mathbb{R}^n$, a system of LMIs is

$$A_0^1 + \theta_1 A_1^1 + \cdots + \theta_n A_n^1 \preccurlyeq 0$$

$$\vdots$$

$$A_0^m + \theta_1 A_1^m + \cdots + \theta_n A_n^m \preccurlyeq 0$$

where $A_0^i, A_1^i, \ldots, A_n^i, i = 1, \ldots, m$, are real symmetric data matrices.

- Affine in decision variables
- Bridge the gap between characteristics of a transfer function and matrix properties of state space models.

Introduction

Learning the dynamics of nonlinear systems with **regional stability guarantees** through LMI constraints.

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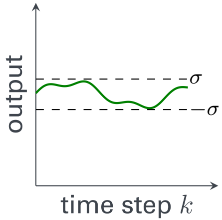
Regional Stability

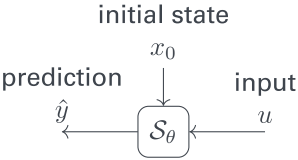
Consider a compact set $\mathcal{X} \subset \mathbb{R}^n$ and an admissible input set $\mathcal{U}$.
The system is **regionally stable** if:

- $\mathcal{X}$ is **forward invariant** for all inputs in $\mathcal{U}$
- All trajectories starting in $\mathcal{X}$ remain in $\mathcal{X}$
- The output trajectories are bounded by $\mathcal{Y}$ for all trajectories starting in $\mathcal{X}$ and all inputs in $\mathcal{U}$

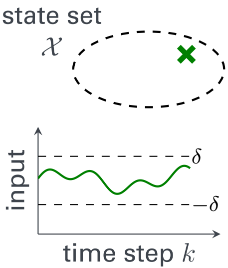
Motivation



output
σ
$-\sigma$
time step k

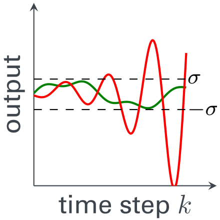


initial state
x_0
prediction
$\hat{y}$
$\mathcal{S}_\theta$
input
u



state set
$\mathcal{X}$
input
δ
$-\delta$
time step k

Motivation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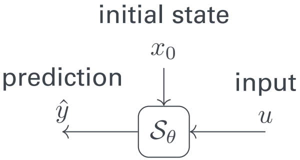
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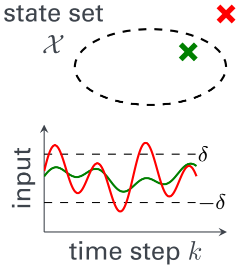
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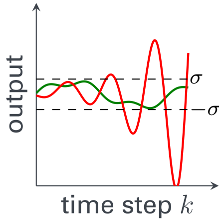

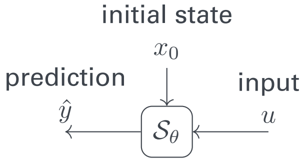

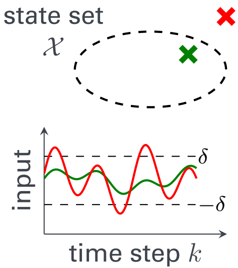

Stability properties cannot be infered from measured data

- Measured trajectories are always bounded
- Stability must be independent of the data

Architecture and Problem Formulation

1

System Identification Setup

Given: Dataset of M input-output trajectories

$$\mathcal{D} = \left\{ \left((u_k, y_k)_{k=0}^{N_i - 1}, x_0 \right)^{[i]} \right\}_{i=1}^{M}$$

with $u_k \in \mathbb{R}^d$ inputs, $y_k \in \mathbb{R}^e$ outputs, and $x_0 \in \mathbb{R}^n$ initial state.

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$$G := \begin{cases} x_{k+1} = Ax_k + Bu_k + B_2 w_k \\ \hat{y}_k = Cx_k + Du_k + D_{12} w_k \\ v_k = C_2 x_k + D_{21} u_k \end{cases}$$

and

$$w_k = \Delta(z_k)$$

with $\Delta = \left(\mathrm{dzn}(v_k^1), \ldots, \mathrm{dzn}(v_k^m) \right)$

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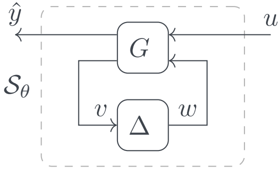

Figure 1: Model architecture

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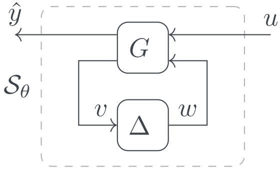

Figure 1: Model architecture

Goal:

Identify a nonlinear state space model $\mathcal{S}_\theta$ that is regionally stable.

Regional stability through LMI constraints

2

Sector-Bounded Nonlinearities

Common activation functions of RNNs are sector-bounded by 0 and 1.

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Standard Sector:

$$\begin{pmatrix} \psi(v) \\ v \end{pmatrix}^\top \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi(v) \\ v \end{pmatrix} \geq 0$$

Properties:

- Graph of the activation function lies **in the sector** defined by lines with slopes 0 and 1

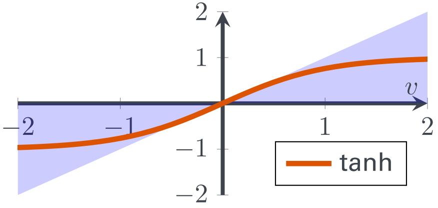

Figure 2: Tanh activation function.

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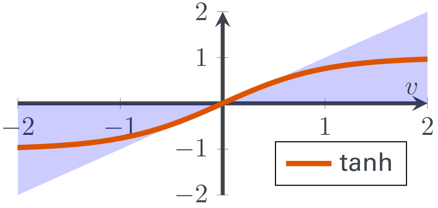

Figure 2: Tanh activation function.

In this work: (based on Tarbouriech et al. [4], La Bella et al. [2])

Change activation function and use *generalized* sector conditions that hold for **deadzone** nonlinearity.

Generalized Sector Conditions

Generalized Sector Function:

$$\begin{pmatrix} \psi(v) \\ v + g \end{pmatrix}^\top \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi(v) \\ v + g \end{pmatrix} \geq 0$$

for $|g| < 1$

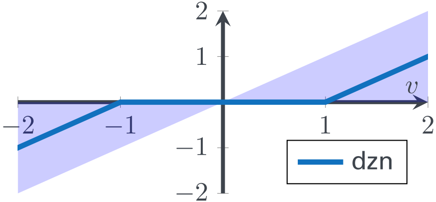

Figure 3: Deadzone activation function.

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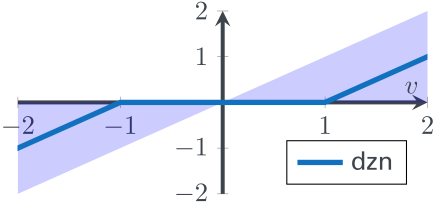
Key difference:

- $g = 0$: Standard sector (global)
- $g \neq 0$: additional degree of freedom



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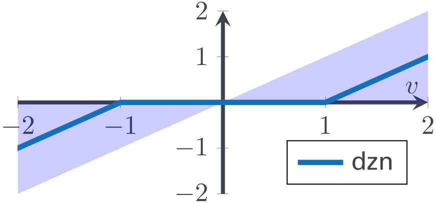

Figure 3: Deadzone activation function.

Connect nonlinearity channel with internal state

$$g = Hx$$

where H is a learnable matrix.
Polyhedron defined by H

$$\mathcal{L}(H) := \{x \in \mathbb{R}^n | \|Hx\|_\infty \leq 1\} \tag{1}$$

Assumptions for regional stability

Initial state condition: $x_0 \in \mathcal{X} = \mathcal{E}(X) := \{x \in \mathbb{R}^n | x^{\mathrm{T}} X x \leq 1\}$

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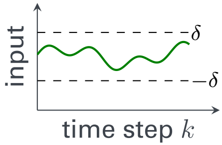
Input condition: $\mathcal{U}_\delta = \left\{(u_k)_{k\in\mathbb{N}} \,\middle|\, u_k^\top u_k \leq \delta^2,\, \forall\, k \in \mathbb{N}\right\}$



Main Result: Regional Input-to-State Stability

Regional input-to-state stability

Suppose there exist matrices $P \in \mathbb{S}_+$, $M \in \mathbb{D}_+$, $L = \begin{pmatrix}(l^1)^\mathrm{T} & \cdots & (l^m)^\mathrm{T}\end{pmatrix}^\mathrm{T} \in \mathbb{R}^{m\times n}$, and scalar $s \in \mathbb{R}$ such that the linear matrix inequalities (LMI)

$$\underbrace{\left(\begin{array}{ccc|c} -\alpha^2 P & 0 & PC_2^\mathrm{T} + L^\mathrm{T} & PA^\mathrm{T} \\ \star & -I & D_{21}^\mathrm{T} & B^\mathrm{T} \\ \star & \star & -2M & MB_2^\mathrm{T} \\ \hline \star & \star & \star & -P \end{array}\right)}_{\mathbf{F}} \preceq 0,\ P \succ 0,\ \text{and}\ \underbrace{\begin{pmatrix} 1/s^2 & l^i \\ (l^i)^\mathrm{T} & P \end{pmatrix}}_{\mathbf{G}_i} \succeq 0,\ \text{for all}\ i = 1,\dots,m, \tag{2}$$

are satisfied. Then, for any input sequence $u \in \mathcal{U}_\delta$ with

$$\delta^2 \le (1-\alpha^2)s^2, \tag{3}$$

the ellipsoid $\mathcal{E}(P^{-1}/s^2)$ is forward invariant and $\mathcal{E}(P^{-1}/s^2) \subset \mathcal{L}(H)$ with $H = LP^{-1}$, $X = P^{-1}$ when $x_0 \in \mathcal{E}(P^{-1}/s^2)$.

State x_k stays in $\mathcal{X}$

Whenever the constraints from above are satisfied then the state set $\mathcal{X}$ is forward invariant

- The internal state of a model is often high dimensional and does not necessarily have a physical meaning.
- State trajectories are usually not available in system identification

Linear mapping from input and state to output

Guaranteed output set

Given a system where the state x_k is strictly bounded by the invariant state ellipsoid $(x_k)^T X x_k \leq s^2$, and an output equation $\hat{y_k} = C x_k$. If there exists a symmetric positive definite matrix $\tilde{Y}$ satisfying the linear matrix inequalitiess (LMIs)

$$\begin{pmatrix} P/s^2 & (CP)^T \\ \star & Y^{-1} \end{pmatrix} \succ 0, \tag{4}$$

and (2), then the output $\hat{y_k}$ is certified to remain bounded within the invariant output ellipsoid defined by $\mathcal{Y} = \mathcal{E}(Y)$ where $Y^{-1} = \tilde{Y}$.

Proof idea: We want to show that

$$y^T Y y \leq 1 \text{ whenever } x^T (X/s^2) x \leq 1$$

for some matrix Y that defines the ellipsoid $\mathcal{E}(Y)$. Or equivalently when $x^k \in \mathcal{X}$ then $y^k \in \mathcal{Y}$ for any $k > 0$.

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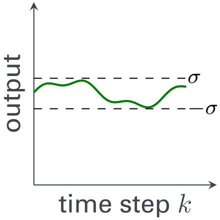

Example

3

Toy example: 2-state nonlinear system

Unknown Data-Generating System:

- 2-state nonlinear system with deadzone nonlinearity
- Regionally stable by construction

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Dataset:

- 900 trajectories, 45 000 datapoints
- Sinusoidal inputs: $u_k = \sqrt{s^2(1-\alpha^2)}\,\sin(kdt)$
- Random inputs: $u \sim \mathcal{R}(-\sqrt{s^2(1-\alpha^2)}, \sqrt{s^2(1-\alpha^2)})$
- Initial conditions: $x_0 \in [-6, 6]$
- **Note:** Some trajectories diverge (outside $\mathcal{X}$)

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Compared Methods:

1. **GenSec**: Our method (generalized sector, regional)
2. **StdSec**: Standard sector (global stability)
3. **NoSec**: No stability constraints

Results: Phase Space Comparison

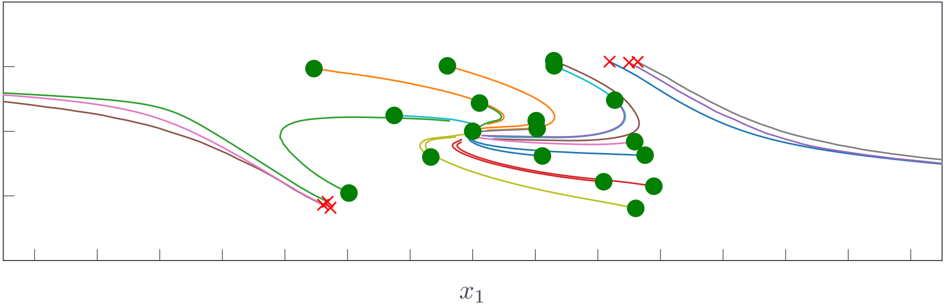


Figure 4: NoSec

Results: Phase Space Comparison

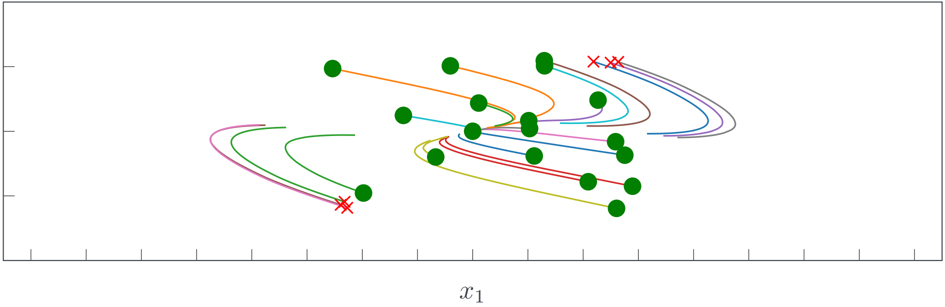


Figure 4: StdSec

Results: Phase Space Comparison

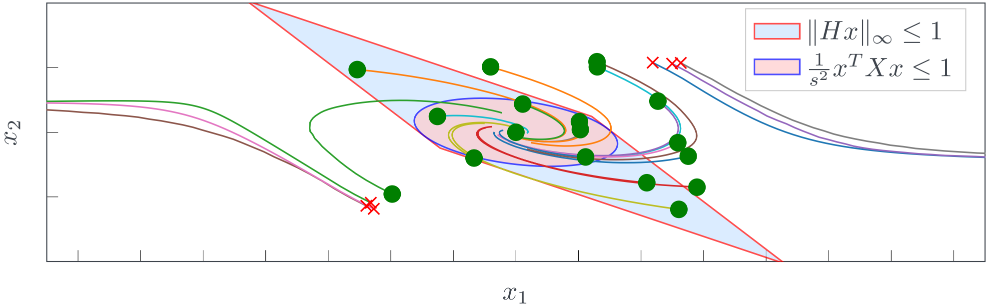
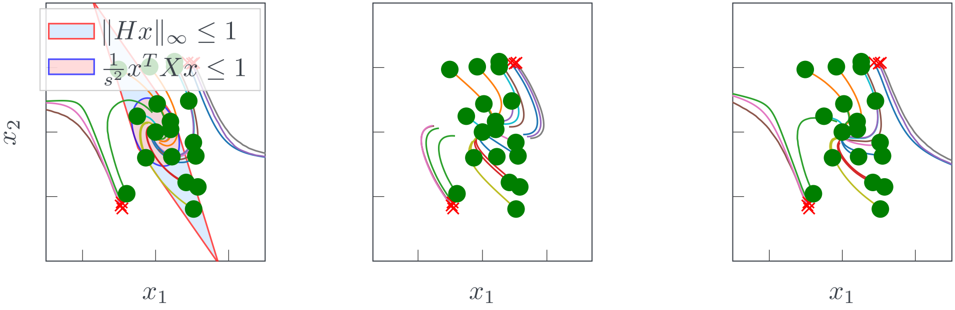


Figure 4: GenSec (Ours)

Results: Phase Space Comparison



(a) GenSec (Ours) (b) StdSec (c) NoSec

Observations:

- **GenSec**: Successfully identifies system with regional guarantees
- **StdSec**: Too conservative, fails to identify dynamics
- **LtiRNN**: No stability guarantees

Results: Quantitative Evaluation

Table 1: Evaluation results on test dataset for the different models.

Model	NRMSE	# params	m
NoSec	0.001581	24	2
StdSec	0.01635	33	2
GenSec (ours)	**0.0005698**	37	2

Silverbox

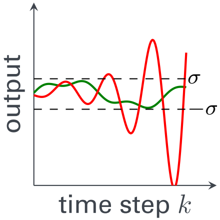
Table 2: Evaluation results on the Silverbox dataset.

Model	$1000\ \mathrm{RMSE}(y, \hat{y})$ in mV	σ in mV	#Par.
GenSec (ours)	[5.72; 9.38; 5.50]	3.39	653
NoSec	[1.67; 2.21; 1.287]	-	553
SUBNET [1]	[0.36; 1.4; 0.32]	-	-

Conclusion

4

Summary



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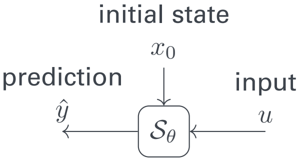

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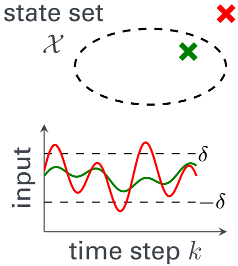

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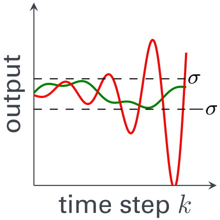

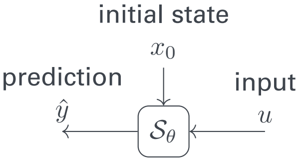

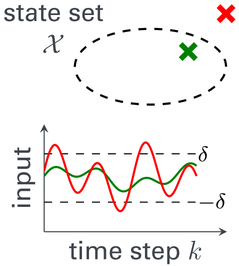

Future Work:

- Reduce conservatism of static input set $\mathcal{U}_\delta$
- Scale to larger neural network architectures

Thank You!

Daniel Frank

University of Stuttgart
Analytic Computing

e-mail: daniel.frank@ki.uni-stuttgart.de
phone: +49 711 685-88107
address: Universitaetsstrasse 32
70569 Stuttgart
room: 2.204

Literature I

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[3] C. Scherer and S. Weiland. Linear matrix inequalities in control. *Lecture Notes, Dutch Institute for Systems and Control, Delft, The Netherlands*, 3(2), 2000.

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Initialization Strategy

Goal: Find initial feasible model

Approach:

1. Fix model parameters θ (e.g., random initialization)
2. Choose α, s based on data properties
3. Solve feasibility SDP for P, L, M:

$$\begin{aligned} \text{find} \quad & P, L, M \\ \text{s.t.} \quad & \text{LMI conditions hold} \end{aligned}$$

Key Advantage:

- Constraints are affine in P, L, M, s^2
- Can use efficient SDP solvers
- Ensures bounded barrier functions at start